III. REDUKSI GANJIL-GENAP/REDUKSI SIKLIS

3.1. Algoritma Sequensial

Contoh 9.2

Selesaikan sistem persamaan berikut :

$$
\begin{aligned}
16x_1 + 4x_2 &= 8 \\
4x_1 + 11x_2 - 5x_3 &= 7 \\
2x_2 + 14x_3 - 6x_4 &= 13 \\
5x_3 + 18x_4 &= 24
\end{aligned}
$$

Jawab

Vektor **x** = $[x_1\ \ x_2\ \ x_3\ \ x_4]^T$ diperoleh melalui prosedur berikut :

1. eliminasi x_1 pada persamaan *kedua* dengan meng-gunakan persamaan *pertama* sehingga diperoleh :

$$
\begin{aligned}
16x_1 + 4x_2 &= 8 \\
10x_2 - 5x_3 &= 5 \\
2x_2 + 14x_3 - 6x_4 &= 13 \\
5x_3 + 18x_4 &= 24
\end{aligned}
$$

2. ulangi langkah (1) untuk x_2 pada persamaan ketiga dan x_3 pada persamaan keempat. Hasilnya adalah :

$$
\begin{aligned}
16x_1 + 4x_2 &= 8 \\
10x_2 - 5x_3 &= 5 \\
15x_3 - 6x_4 &= 12 \\
20x_4 &= 20
\end{aligned}
$$

3. selesaikan persamaan terakhir, dimulai dari penyelesaian x_4 pada persamaan keempat.

Dengan cara ini maka diperoleh :

x = $[x_1\ \ x_2\ \ x_3\ \ x_4]^T = [0.225\ \ 1.100\ \ 1.200\ \ 1.000]^T$

Sistem di atas dapat dituliskan sebagai **A x** = **b**, dimana :

$$\mathbf{A} = \begin{bmatrix} 16 & 4 & & \\ 4 & 11 & -5 & \\ & 2 & 14 & -6 \\ & & 5 & 18 \end{bmatrix} = \text{matriks tridiagonal},\ \ \mathbf{b} = \begin{bmatrix} 8 \\ 7 \\ 13 \\ 24 \end{bmatrix}$$

Untuk sistem tridiagonal berukuran *n* prosedur di atas dapat digeneralisasi menjadi algoritma sekuensial berikut :

Algoritma 9.3. TRIDIAGONAL.SYSTEM.SOLVER (SISD) :

```
  Global   n                                {ukuran sistem}
            f[2.. n], g[1.. n], h[1..( n -1)]  {elemen matriks A}
           b[1.. n]                         {elemen vektor b}
           x[1.. n]                         {elemen vektor x}
begin
   for i ← 1 to n –1 do
      g[i+1] ← g[i+1] – (f[i+1] / g[i]) x h[i]
      b[i+1] ← b[i+1] – (f[i+1] / g[i]) x b[i]
   endfor
   for i ← n downto 2 do
      x[i] ← b[i] / g[i]
      b[i –1] ← b[i –1] – x[i] x h[i –1]
   endfor
   x[1] ← b[1] / g[1]
end
```

Algoritma 9.3. mengandung 2 loop masing-masing dengan *n–1* iterasi. Setiap iterasi memerlukan waktu yang tetap. Kompleksitas algoritma dengan demikian adalah Θ(*n*).

3.2. Potensi Paralelisasi

Perhatikan dua matriks berikut :

$$\mathbf{A} = \begin{bmatrix} g_1 & h_1 & & \\ f_2 & g_2 & h_2 & \\ & f_3 & g_3 & h_3 \\ & & f_4 & g_4 \end{bmatrix},\quad \mathbf{A}^{(1)} = \begin{bmatrix} g_1^{(1)} & h_1^{(1)} & & \\ f_2^{(1)} & g_2^{(1)} & 0 & \\ & 0 & g_3^{(1)} & h_3^{(1)} \\ & & f_4^{(1)} & g_4^{(1)} \end{bmatrix}$$

Matriks $\mathbf{A}^{(1)}$ adalah hasil transformasi dari matriks **A**. Matriks $\mathbf{A}^{(1)}$ terdiri dari dua matriks tridiagonal yang saling lepas sehingga penyelesaian kedua bagian vektor **x** dapat dilakukan secara serempak.

3.3. Algoritma Paralel

Algoritma paralel dibangun untuk mentransformasikan matriks **A** menjadi $\mathbf{A}^{(1)}$. Pola umum struktur matriks tridiagonal berukuran *n* adalah :

$$
\begin{aligned}
&g_1 x_1 + h_1 x_2 && = b_1 \\
&f_i x_{i-1} + g_i x_i + h_i x_{i+1} && = b_i \qquad 2 \le i \le n\text{-}1 \qquad (9.4)\\
&f_n x_{n-1} + g_n x_n && = b_n
\end{aligned}
$$

Untuk $3 \le i \le n\text{-}2$, persamaan (*i-1*), (*i*), dan (*i+1*) adalah :

$$
\begin{aligned}
&f_{i-1}\ x_{i-2} + \boxed{g_{i-1}\ x_{i-1}} + h_{i-1}\ x_i && = b_{i-1}\\
&\qquad\quad \boxed{f_i\, x_{i-1}} + g_i\, x_i + h_i\, x_{i+1} && = b_i\\
&\qquad\qquad\qquad f_{i+1}\ x_i + \boxed{g_{i+1}\ x_{i+1}} + h_{i+1}\ x_{i+2} && = b_{i+1}
\end{aligned}
$$

Dari persamaan (*i-1*) dan (*i+1*) dapat diperoleh :

$$x_{i-1} = (1/g_{i-1})(b_{i-1} - f_{i-1}\ x_{i-2} - h_{i-1}\ x_i) \qquad (9.4a)$$

$$x_{i+1} = (1/g_{i+1})(b_{i+1} - f_{i+1}\ x_i - h_{i+1}\ x_{i+2}) \qquad (9.4b)$$

Substitusi kedua persamaan ke persamaan (*i*) akan menghasilkan :

$$f_i^{(1)}\ x_{i-2} + g_i^{(1)}\ x_i + h_i^{(1)}\ x_{i+2} = b_i^{(1)} \qquad (9.5)$$

dimana :

$$f_i^{(1)} = \alpha_i^{(1)}\ f_{i-1}, \quad g_i^{(1)} = g_i + \alpha_i^{(1)}\ h_{i-1} + \beta_i^{(1)}\ f_{i+1} \qquad (9.6a)$$

$$h_i^{(1)} = \beta_i^{(1)}\ h_{i+1}, \quad b_i^{(1)} = b_i + \alpha_i^{(1)}\ b_{i-1} + \beta_i^{(1)}\ b_{i+1} \qquad (9.6b)$$

$$\alpha_i^{(1)} = -\frac{f_i}{g_{i-1}}, \qquad \beta_i^{(1)} = -\frac{h_i}{g_{i+1}} \qquad (9.6c)$$

Persamaan (9.6) juga berlaku untuk : $f_{n-1}^{(1)}$, $f_n^{(1)}$, $g_2^{(1)}$, $g_{n-1}^{(1)}$, $h_1^{(1)}$, $h_2^{(1)}$, kecuali untuk :

$$g_1^{(1)} = g_1 + \beta_1^{(1)}\ f_2 \quad \text{dan} \quad g_n^{(1)} = g_n + \alpha_n^{(1)}\ h_{n-1} \qquad (9.6d)$$

$$b_1^{(1)} = b_1 + \beta_1^{(1)}\ b_2 \quad \text{dan} \quad b_n^{(1)} = b_n + \alpha_n^{(1)}\ b_{n-1} \qquad (9.6e)$$

Ungkapan matriks persamaan (9.5) adalah $\mathbf{A}^{(1)}\mathbf{x} = \mathbf{b}^{(1)}$ yang merupakan transformasi dari $\mathbf{Ax} = \mathbf{b}$ dimana :

$$
\mathbf{A}^{(1)} = \begin{bmatrix}
g_1^{(1)} & 0 & h_1^{(1)} & & & \\
0 & g_2^{(1)} & 0 & h_2^{(1)} & & \\
f_3^{(1)} & 0 & g_3^{(1)} & \mathrm{O} & \mathrm{O} & \\
 & f_4^{(1)} & \mathrm{O} & \mathrm{O} & \mathrm{O} & h_{n-2}^{(1)} \\
 & & \mathrm{O} & \mathrm{O} & g_{n-1}^{(1)} & 0 \\
 & & & f_n^{(1)} & 0 & g_n^{(1)}
\end{bmatrix} \qquad (9.7)
$$

Persamaan $\mathbf{A}^{(1)}\mathbf{x} = \mathbf{b}^{(1)}$ dapat disusun ulang menjadi $\overline{\mathbf{A}}^{(1)}\overline{\mathbf{x}} = \overline{\mathbf{b}}^{(1)}$, dimana $\overline{\mathbf{A}}^{(1)}$ =

$$
\begin{bmatrix}
g_1^{(1)} & h_1^{(1)} & & & & & & & & \\
f_3^{(1)} & g_3^{(1)} & h_3^{(1)} & & & & & & & \\
 & f_5^{(1)} & \mathrm{O} & \mathrm{O} & & & & & & \\
 & & \mathrm{O} & \mathrm{O} & h_{n-3}^{(1)} & & & & & \\
 & & & f_{n-1}^{(1)} & g_{n-1}^{(1)} & 0 & & & & \\
 & & & & 0 & g_2^{(1)} & h_2^{(1)} & & & \\
 & & & & & f_4^{(1)} & g_4^{(1)} & h_4^{(1)} & & \\
 & & & & & & f_6^{(1)} & \mathrm{O} & \mathrm{O} & \\
 & & & & & & & \mathrm{O} & \mathrm{O} & h_{n-2}^{(1)} \\
 & & & & & & & & f_n^{(1)} & g_n^{(1)}
\end{bmatrix} \qquad (9.8a)
$$

$$
\overline{\mathbf{x}} = [\,x_1 \;\; x_3 \;\; \dots \;\; x_{n-1} \;\; x_2 \;\; x_4 \;\; x_n\,]^T \qquad (9.8b)
$$

$$
\overline{\mathbf{b}}^{(1)} = [\,b_1^{(1)} \;\; b_3^{(1)} \;\; b_{n-1}^{(1)} \;\; b_2^{(1)} \;\; b_4^{(1)} \;\; b_n^{(1)}\,]^T \qquad (9.8c)
$$

Berdasarkan pembahasan di atas maka dapat disusun sebuah algoritma seperti berikut di bawah ini.

Algoritma 9.4. CYCLIC.REDUCTION(SIMD) :

Global n	{ukuran sistem}
$f[2..n]$, $g[1..n]$, $h[1..(n-1)]$	{elemen **A**}
$b[1..n]$	{elemen **b**}
$f^{(1)}[3..n]$, $g^{(1)}[1..n]$, $h^{(1)}[1..(n-2)]$	{elemen $\mathbf{A}^{(1)}$}
$b^{(1)}[1..n]$	{elemen $\mathbf{b}^{(1)}$}
p	{# proses}
$\alpha^{(1)}[2..n]$, $\beta^{(1)}[1..(n-1)]$	{faktor skala}
$x[1..n]$	{elemen vektor **x**}

begin
  for $i \leftarrow 1$ to $n-1$ do
    $\alpha^{(1)}[i+1] \leftarrow -f[i+1] / g[i]$
    $\beta^{(1)}[i] \leftarrow -h[i]/ g[i+1]$
  endfor
  for $i \leftarrow 2$ to $n-1$ do
    $f^{(1)}[i+1] \leftarrow \alpha^{(1)}[i+1] \times f[i]$
    $g^{(1)}[i] \leftarrow g[i] + \alpha^{(1)}[i] \times h[i-1] + \beta^{(1)}[i] \times f[i+1]$
    $h^{(1)}[i-1] \leftarrow \beta^{(1)}[i-1] \times h[i]$
    $b^{(1)}[i] \leftarrow b[i] + \alpha^{(1)}[i] \times b[i-1] + \beta^{(1)}[i] \times b[i+1]$
  endfor
  $g^{(1)}[1] \leftarrow g[1] + \beta^{(1)}[1] \times f[2]$
  $g^{(1)}[n] \leftarrow g[n] + \alpha^{(1)}[n] \times h[n-1]$
  $b^{(1)}[1] \leftarrow b[i] + \beta^{(1)}[1] \times b[2]$
  $b^{(1)}[n] \leftarrow b[n] + \alpha^{(1)}[n] \times b[n-1]$
  forall P[j] where $1 \le j \le 2$ do
    for $i \leftarrow 2$ to $n-1$ step 2 do
      $g^{(1)}[i+j] \leftarrow g^{(1)}[i+j]$
          $-(f^{(1)}[i+j] / g^{(1)}[i+j-2])$ x $h^{(1)}[i+j-2]$
      $b^{(1)}[i+j] \leftarrow b^{(1)}[i+j]$
          $-(f^{(1)}[i+j] / g^{(1)}[i+j-2])$ x $b^{(1)}[i+j-2]$
    endfor
    for $i \leftarrow n-2+j$ downto $j+1$ do step 2
      $x[i] \leftarrow b^{(1)}[i] / g^{(1)}[i]$
      $b^{(1)}[i-2] \leftarrow b^{(1)}[i-2] - x[i]$ x $h^{(1)}[i-2]$
    endfor
    $x[j] \leftarrow b^{(1)}[j] / g^{(1)}[j]$
  endfor
end

Algoritma 9.4. terdiri dari 4 loop masing-masing dengan kompleksitas algoritma $\Theta(n)$. Berikut ini adalah *tracing* penerapan algoritma 9.4. terhadap sistem persamaan pada contoh 9.2. :

looping 2-5 :

$i = 1 : \alpha^{(1)}[2] = -f[2] / g[1] = -4/16 = -1/4$

$\beta^{(1)}[1] = - h[1] / g[2] = -4/11$

$i = 2 : \alpha^{(1)}[3] = -f[3] / g[2] = -2/11$

$\beta^{(1)}[2] = - h[2] / g[3] = 5/14$

$i = 3 : \alpha^{(1)}[4] = -f[4] / g[3] = -5/14$

$\beta^{(1)}[3] = - h[3] / g[4] = 6/18 = 1/3$

looping 6 – 11 :

$i = 2 : f^{(1)}[3] = \alpha^{(1)}[3] \times f[2] = -2/11 \times 4 = -8/11$

$g^{(1)}[2] = g[2] + \alpha^{(1)}[2] \times h[1] + \beta^{(1)}[2] \times f[3]$

$= 11 - 1/4 \times 4 + 5/14 \times 2 = 75/7$

$h^{(1)}[1] = \beta^{(1)}[1] \times h[2] = -4/11 \times (-5) = 20/11$

$b^{(1)}[2] = b[2] + \alpha^{(1)}[2] \times b[1] + \beta^{(1)}[2] \times b[3]$

$= 7 - 1/4 \times 8 + 5/14 \times 13 = 135/14$

$i = 3 : f^{(1)}[4] = \alpha^{(1)}[4] \times f[3] = -5/14 \times 2 = -5/7$

$g^{(1)}[3] = g[3] + \alpha^{(1)}[3] \times h[2] + \beta^{(1)}[3] \times f[4]$

$= 14 - 2/11 \times (-5) + 1/3 \times 5 = 547/33$

$h^{(1)}[2] = \beta^{(1)}[2] \times h[3] = 5/14 \times (-6) = -15/7$

$b^{(1)}[3] = b[3] + \alpha^{(1)}[3] \times b[2] + \beta^{(1)}[3] \times b[4]$

$= 13 - 2/11 \times 7 + 1/3 \times 24 = 217/11$

baris 12-15 :

$g^{(1)}[1] = g[1] + \beta^{(1)}[1] \times f[2] = 16 - 4/11 \times 4 = 160/11$

$g^{(1)}[4] = g[4] + \alpha^{(1)}[4] \times h[3] = 18 - 5/14 \times (-6) = 141/7$

$b^{(1)}[1] = b[1] + \beta^{(1)}[1] \times b[2] = 8 - 4/11 \times 7 = 60/11$

$b^{(1)}[4] = b[4] + \alpha^{(1)}[4] \times b[3] = 24 - 5/14 \times 13 = 271/14$

Sampai tahap ini terbentuk matriks dan vektor berikut :

$$\begin{bmatrix} g_1^{(1)} & h_1^{(1)} & & \\ f_3^{(1)} & g_3^{(1)} & 0 & \\ & 0 & g_2^{(1)} & h_2^{(1)} \\ & & f_4^{(1)} & g_4^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{160}{11} & \frac{20}{11} & & \\ \frac{-8}{11} & \frac{547}{33} & 0 & \\ & 0 & \frac{75}{7} & \frac{-15}{7} \\ & & \frac{-5}{7} & \frac{141}{7} \end{bmatrix}, \quad \begin{bmatrix} b_1^{(1)} \\ b_3^{(1)} \\ b_2^{(1)} \\ b_4^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{60}{11} \\ \frac{217}{11} \\ \frac{135}{14} \\ \frac{271}{14} \end{bmatrix}$$

looping paralel 16 – 28, j = 1 :
sublooping 17-20

$i = 2 : \quad g^{(1)}[3] = g^{(1)}[3] - (f^{(1)}[3] / g^{(1)}[1]) \times h^{(1)}[1]$

$= (547/33) - ((-8/11) / (160/11)) \times 20/11 = 550/33$

$b^{(1)}[3] = b^{(1)}[3] - (f^{(1)}[3] / g^{(1)}[1]) \times b^{(1)}[1]$

$= (217/11) - ((-8/11) / (160/11)) \times 60/11 = 220/11$

sublooping 21-24

i = 3 $\quad x[3] = b[3] / g[3] = (220/11) / (550/33) = \boxed{\mathbf{1.200}}$

$\qquad b[1] = b[1] - x[3] \times h[1] = 60/11 - 1.200 \times 20/11 = 36/11$

instruksi 25 :

$\qquad x[1] = b[1] / g[1] = (36/11) / (160/11) = \boxed{\mathbf{0.225}}$

looping paralel 16 – 28, j = 2 :

sublooping 17-20

i = 2 : $\quad g^{(1)}[4] = g^{(1)}[4] - (f^{(1)}[4] / g^{(1)}[2]) \times h^{(1)}[2]$

$\qquad = (141/7) - ((-5/7) / (75/7)) \times (-15/7) = 20$

$\qquad b^{(1)}[4] = b^{(1)}[4] - (f^{(1)}[4] / g^{(1)}[2]) \times b^{(1)}[2]$

$\qquad = (271/14) - ((-5/7) / (75/7)) \times 135/14 = 20$

sublooping 21-24

i = 3 $\quad x[4] = b[4] / g[4] = 20 / 20 = \boxed{\mathbf{1.000}}$

$\qquad b[2] = b[2] - x[4] \times h[2] = 135/14 - 1.000 \times (-15/7) = 165/14$

instruksi 25 :

$\qquad x[2] = b[2] / g[2] = (165/14) / (75/7) = \boxed{\mathbf{1.100}}$

2.4. Speedup

Kompleksitas algoritma dapat juga dinyatakan melalui nilai total operasi floating point. Untuk algoritma 9.3. nilai total operasi floating point adalah :

$$\sum_{i=1}^{n-1} 6 + \sum_{i=2}^{n-1} 3 + 1 = 9n - 8 \qquad (9.9)$$

sedangkan untuk algoritma 9.4. adalah :

$$\sum_{i=1}^{n-1} 2 + \sum_{i=2}^{n-1} 10 + 8 + \sum_{i=2}^{n-1} 6/2 + \sum_{i=1}^{n-2} 3/2 + 1 = 16.5n - 22 \qquad (9.10)$$

sehingga speedup adalah :

$$\lim_{n \to \infty} \frac{9n - 8}{16.5n - 20} = 0.545 \qquad (9.11)$$

Jika paralelisasi juga dilakukan pada langkah 2-5, 6-11, dan 12-16 maka total operasi floating point algoritma 9.4 adalah :

$$\sum_{i=1}^{n-1} 2/2 + \sum_{i=2}^{n-1} 10/2 + 8/2 + \sum_{i=2}^{n-1} 6/2 + \sum_{i=1}^{n-2} 3/2 + 1 = 10.5n - 18 \qquad (9.12)$$

sehingga speedup adalah :

$$\lim_{n \to \infty} \frac{9n - 8}{10.5n - 10} = 0.857 \qquad (9.13)$$

Baik (9.11) maupun (9.12) adalah nilai speedup yang buruk (< 1.0). Kedua nilai tersebut dihasilkan jika paralelisasi menggunakan 2 prosesor. Jika menggunakan 4 prosesor maka instruksi-instruksi yang dapat di-*assign* ke masing-masing prosesor adalah instruksi : 6-10, 12-15, dan, dengan sedikit perbedaan penanganan, langkah 16-25. Total operasi floating point dengan 4 prosesor adalah :

$$\sum_{i=1}^{n-1} 1 + \sum_{i=2}^{n-1} 4 + 2 + \sum_{i=2}^{n-1} 3/2 + \sum_{i=1}^{n-2} 2/2 + 1 = 7.5n - 14 \qquad (9.14)$$

sehingga speedup adalah :

$$\lim_{n \to \infty} \frac{9n - 8}{7.5n - 14} = 1.2 \qquad (9.15)$$

Nilai (9.15) ini adalah nilai maksimum untuk paralelisasi algoritma 9.4.